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LETTER TO THE EDITOR

Replica symmetry breaking in finite connectivity systems: a large connectivity expansion at finite and zero temperature

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Abstract. Parisi replica symmetry breaking is extended to random bond systems with *finite*, fixed connectivity $(M+1)$. The free energy f is explicitly calculated for *large* M within the first stage of symmetry beaking. At *finite* temperature T, the $1/M$ expansion is found to diverge as $T \rightarrow 0$. Indeed, direct evaluation at *zero* temperature shows that the large M expansion is in powers of $1/\sqrt{M}$. Introduction of symmetry breaking brings the value of f close ($\sim 1\%$ for $10 < M < 20$) to numerical estimates of Banavar *et al.* The same techniques apply to systems with average *finite* connectivity.

1. Introduction

Much interest has recently been devoted to the theory of randomly frustrated systems (particularly spin glasses) on lattices with *finite* connectivity. Indeed those systems are closer in nature to real spin glasses because of the finite valence nature of the lattice. Besides, such systems are directly connected to some well known optimisation problems (graph partitioning, colouring, etc,). Nevertheless they preserve many of the simplifying features of a mean-field theory because small loops are very improbable.

Previous treatments of such systems, except for the immediate vicinity of T_c, the transition temperature, or exactly soluble models§, have been confined within the framework of replica symmetry (i.e. assuming a single thermodynamic state) although evidence has been accumulating (Viana and Bray 1985, Mottishaw 1987, Mottishaw and De Dominicis 1987, Goldschmidt 1988a, de Almeida *et al* 1988) that replica symmetry (RS) has to be broken in many of the cases under consideration.

In order to go beyond and evaluate precisely the free energy of the systems both at *finite* and *zero* temperature, we have used an *expansion in* $1/M$ where $M+1$ *is the connectivity.* This is the analogue of the $1/d$ expansion for real lattices. The method, at least at finite T, can be extended to hypercubic lattices, but this problem will not be pursued further here‖.

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§ That is, when the random bond is not between doublets but between p-plets, $p \rightarrow \infty$ as in the model of Derrida (1980). This case is treated by De Dominicis and Mottishaw (1987c).

‖ Upon completion of this work we learned that George, Mézard and Yedidia, at the ENS, are working on the $1/d$ expansion for the spin glass on the hypercubic lattice at finite temperature.

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In this paper we consider two models:

(i) random lattices with *average finite connectivity*, with a bond distribution as in Viana and Bray,

$$\mathcal{P}(J)=\left(1-\frac{\alpha}{N}\right)\delta(J)+\frac{\alpha}{N}\rho(J)$$

where $\rho(J)$ is normalised;

(ii) random lattices with *fixed connectivity* $\alpha=M+1$, and a bond distribution $\rho(J)$.

As an alternative to model (ii) one may consider the spin glass on a Bethe lattice but in this case boundary conditions play an important role †.

For these models it is possible to systematically develop *successive stages of* RS *breaking* both at *finite* and *zero temperature* in the *large α or M expansion.*

Model (ii) is more closely related to hypercubic lattices (Katsura (1986) has shown that the equations involved are identical to the Bethe approximation on such a lattice). Also in this case, there exist numerical results with which we can make comparison. For these reasons, we first concentrate on this *fixed* connectivity model: RS breaking is considered at finite T in § 2, and at zero T in § 3. The 1/M expansion is found to diverge as $T\to 0$, signalling what is found by working directly at $T=0$, namely that it really is a $1/\sqrt{M}$ expansion. Comparison with numerical results shows improvement when RS breaking is introduced. Finally in § 4 we indicate how to treat model (i) of *average* finite connectivity.

2. Fixed connectivity finite temperature

Mottishaw has shown that the system can be described by the global order parameter $g(\{\sigma_\alpha\})$ (De Dominics and Mottishaw 1986, 1987a, b) satisfying

$$g_n(\{\sigma_\alpha\})=\int \mathrm{d}J\,\rho(J)\operatorname*{Tr}_{\tau_\alpha} g_n^M(\{\tau_\alpha\})\exp\left(\beta J\sum_\alpha \sigma_\alpha\tau_\alpha\right)\left(\operatorname*{Tr}_{\tau_\alpha} g_n^M(\{\tau_\alpha\})\right)^{-1} \tag{1}$$

where σ_α, $\alpha=1,2,\ldots,n$, are the replicated spin variables and $M+1$ is the connectivity‡. At finite temperature g_n can be parametrised in the form

$$g_n(\{\sigma_\alpha\})=\sum_{r=0}^{\infty} b_r \sum_{(\alpha_1\ldots\alpha_r)} q_{\alpha_1\ldots\alpha_r}\sigma_{\alpha_1}\ldots\sigma_{\alpha_r} \tag{2}$$

with

$$b_r=\langle\cosh^n\beta J\tanh^r\beta J\rangle \tag{3}$$

the averaging being with respect to $\rho(J)$. In the rest of the paper we use for simplicity

$$\rho(J)=\tfrac{1}{2}(\delta(J+J_0)+\delta(J-J_0)) \tag{4}$$

† The random lattice is locally similar to the Bethe lattice (small loops are rare) but it has no boundaries whereas the Bethe lattice does. It has been shown (Chayes *et al* 1986, Carlson *et al* 1988, Lai and Goldschmidt 1989) that some boundary conditions (e.g. fixed ones) on the Bethe lattice project out only a single thermodynamic state. However, there is numerical evidence (Dewar and Mottishaw 1988, Lai and Goldschmidt 1989) that closing the lattice by identifying different points on the boundary allows many states to coexist and leads to a non-trivial overlap function $P(q)$.

‡ Note that our normalisation of g_n differs from that of Mottishaw (1987) for finite n.

(in the averaged quantities we shall omit the index zero in J_0). It is a feature of *finite* connectivity that one has to consider all $q_{\alpha_1\ldots\alpha_r}$ (whereas in the Sherrington-Kirkpatrick (Sherrington and Kirkpatrick 1975) limit, as $M\to\infty$ only $q_{\alpha_1\alpha_2}$ occurs) which satisfy

$$q_{\alpha_1\ldots\alpha_r}=\operatorname*{Tr}_{\sigma_\alpha} g^M(\{\sigma_\alpha\})\sigma_{\alpha_1}\ldots\sigma_{\alpha_r}\left(\operatorname*{Tr}_{\sigma_\alpha} g^M(\{\sigma_\alpha\})\right)^{-1}. \tag{5}$$

We have been able to express the stationary free energy density in terms of $g(\{\sigma_\alpha\})$:

$$\beta f n = M \ln \operatorname*{Tr}_{\sigma_\alpha} g_n^{M+1}(\{\sigma_\alpha\})$$

$$-\frac{M+1}{2}\ln\left[\int \mathrm{d}J\,\rho(J)\operatorname*{Tr}_{\sigma_\alpha}\operatorname*{Tr}_{\tau_\alpha} g_n^M(\{\sigma_\alpha\})g_n^M(\{\tau_\alpha\})\exp\left(\beta J\sum_\alpha \sigma_\alpha\tau_\alpha\right)\right]. \tag{6}$$

This free energy is *independent* of the normalisation of g_n. As usual the limit $n\to 0$ is to be taken. Variation of (6) with respect to $g_n(\{\sigma_\alpha\})$ yields (1), up to a normalisation of g_n.

The $1/M$ expansion is derived by scaling the coupling $J=\tilde{J}/\sqrt{M}$ and considering large M values. This can be done by building g^M from (1) taken to the mth power or using (2) that, together with (3) and (4) gives, using for shorthand $\lambda\equiv\beta\tilde{J}$,

$$g_n^M=(\cosh^n(\lambda/\sqrt{M}))^M\left[1+\frac{\lambda^2}{M}\left(1-\frac{2\lambda^2}{3M}\right)\sum_{(\alpha,\beta)} q_{\alpha\beta}\sigma_\alpha\sigma_\beta\right.$$

$$\left.+\frac{\lambda^4}{M^2}\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\delta}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta+\ldots\right]^M \tag{7}$$

and hence

$$g_n^M(\{\sigma_\alpha\})=\exp\left(\lambda^2\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta\right)(1+\mathrm{O}(1/M)) \tag{8}$$

where $q_{\alpha\beta}^{(0)}$ is the leading term in $q_{\alpha\beta}$ in the $1/M$ expansion.

The full expression for g^M will be reported elsewhere (Goldschmidt and De Dominicis 1989). Substituting the expression for g^M in the free energy (6) and using (5) we get the free energy density $\beta f\equiv\beta f_0+\beta f_1/M+\mathrm{O}(1/M^2)$ with

$$\beta f_0=-\frac{\lambda^2}{4}+\frac{\lambda^2}{2n}\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)^2}-\frac{1}{n}\ln\operatorname{Tr}\exp\sum_{(\alpha\beta)}\lambda^2 q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta$$

$$\beta f_1=-\frac{\lambda^2}{4}+\frac{\lambda^4}{24}-\frac{\lambda^2}{2n}\left(1-\frac{5\lambda^2}{3}\right)\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)^2}-\frac{\lambda^4}{2n}\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\delta}^{(0)^2} \tag{9}$$

$$+\frac{3\lambda^4}{n}\sum_{(\alpha\beta\gamma)} q_{\alpha\beta}^{(0)}q_{\beta\gamma}^{(0)}q_{\gamma\alpha}^{(0)}+\frac{\lambda^4}{n}\sum_{(\alpha\beta\gamma\delta)}(q_{\alpha\beta}^{(0)}q_{\gamma\delta}^{(0)}+2\text{ perm})q_{\alpha\beta\gamma\delta}^{(0)}.$$

Here βf_0 coincides with the term derived by Parisi (1980) in the infinite-ranged model and $\beta f_1/M$ is the first correction due to *finite* connectivity. Note that (9) is no longer stationary since we have made use of the equation of motion. Note also that the q with higher numbers of indices will occur at higher order in the $1/M$ expansion.

(i) RS case. It is easy to evaluate (9) in this case. In figure 1 we plot the free energy density $f/J\sqrt{M}$ against $T/J\sqrt{M}$ for $M=\infty$ and $M=10$. We see that the $1/M$ correction is well behaved up to $1/J\sqrt{10}\sim 0.2$ (where $T_c/J\sqrt{10}=1.0355$); afterwards it appears to diverge. We have verified analytically by considering the corrections to q_2 and q_4 away from zero temperature that indeed the $1/M$ correction diverges as $\beta\to\infty$. Below we will see that this phenomenon occurs because at $T=0$ the expansion in large connectivity is in powers of $1\sqrt{M}$ instead of $1/M$. Nevertheless the extrapolation of the finite temperature result to zero temperature yields results consistent with those obtained in § 3, directly at $T=0$.

(ii) RS breaking: Since it has been shown (Mottishaw) that the RS solution is unstable, we have to break the symmetry. Near T_c one can introduce continuous Parisi order parameter functions $q_2(x)$, $q_4(x, y, z)$, etc, and solve in powers of $T-T_c$. In the entire temperature range we can obtain a solution up to a given stage of RS breaking. For example, within the *first stage of* RS *breaking* one defines $\alpha\equiv(K,\gamma)$, $K=1,2,\ldots,n/m$, and $\gamma=1,2,\ldots,m$. One classifies the values of $q_{\alpha_1\ldots\alpha_r}$ according to the number of spin indices in the same box K, e.g. for $q_{\alpha_1\alpha_2}$ there are two values q_2 and q_{11} (referring to one K box with two spins, and two K boxes with one spin, respectively), for $q_{\alpha_1\alpha_2\alpha_3\alpha_4}$ there are five values q_4, q_{22}, q_{31}, q_{211}, q_{1111} (De Dominicis and Mottishaw 1987c). In this case the global order parameter depends on the spins through $\sigma_K\equiv\Sigma_\gamma\,\sigma_{K\gamma}$ and satisfies

$$g_n\{\sigma_K/\beta\}=\mathcal{N}^{-1}\int \mathrm{d}J\,\rho(J)\int\prod_K\frac{\mathrm{d}s_K}{2\pi}\,g_n^M\{\mathrm{i}s_K/\beta\}$$
$$\times\int\prod_K \mathrm{d}u_K\,\exp(\mathrm{i}s_Ku_K)\exp\left(-\frac{\sigma_K}{\beta}\tanh^{-1}(\tanh\beta J\tanh\beta u_K)\right)$$
$$\times\exp[(m/2)\ln(\cosh^2\beta J\cosh^2\beta u_K-\sinh^2\beta J\sinh^2\beta u_K)] \qquad (10)$$

where the normalisation $\mathcal{N}$ is the same integral with $J=0$ in the exponents.

We found it convenient to introduce the effective field distribution $P_n^{(M)}\{h_K\}$ Fourier transform of $g_n^M\{is_K/\beta\}$. We used it to evaluate the traces in equation (5). For example

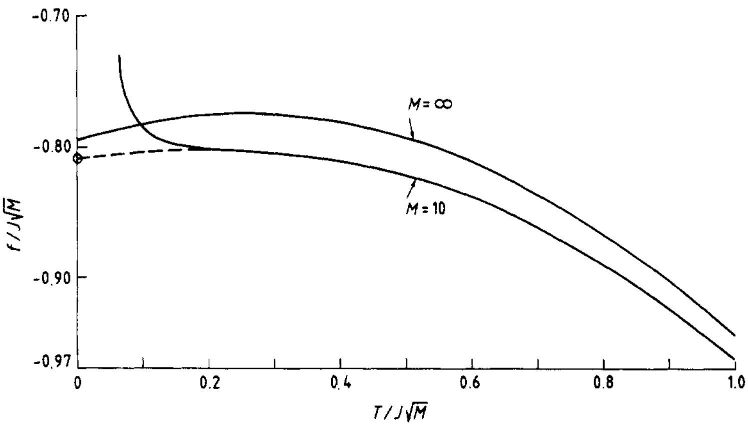

Figure 1. Plot of the rescaled free energy against the rescaled temperature for $M=\infty$ (SK model) and $M=10$ with no RS breaking. The broken line is the extrapolation to zero temperature. The value we obtain directly at $T=0$ is encircled.

$q_2^{(0)}$ satisfies the equation

$$q_2^{(0)}=\int_{-\infty}^{\infty}\frac{\mathrm{d}H}{\sqrt{2\pi q_{11}^{(0)}}}\exp\left(-\frac{H^2}{2q_{11}^{(0)}}\right)$$
$$\times\int_{-\infty}^{\infty}\mathrm{d}h\,\exp\left(-\frac{(h-H)^2}{2(q_2^{(0)}-q_{11}^{(0)})}\right)\cosh^m\beta h\,\tanh^2\beta h$$
$$\times\left[\int_{-\infty}^{\infty}\mathrm{d}h\,\exp\left(-\frac{(h-H)^2}{2(q_2^{(0)}-q_{11}^{(0)})}\right)\cosh^m\beta h\right]^{-1} \tag{11}$$

and similarly for $q_{11}^{(0)}$†. The order parameters $q_4^{(0)}$, $q_{22}^{(0)}$, etc, are given by similar formulae but with only q_2^0, q_{11}^0 appearing on the right-hand side.

We have evaluated the free energy f (9) to first-stage RS breaking. The value of m is then fixed by extremising f. Details will be presented elsewhere (Goldschmidt and De Dominicis 1989).

3. Fixed connectivity zero temperature

Let us define

$$\gamma_n\{x_K\}\equiv g_n\{\sigma_K/\beta\} \tag{12}$$

and consider (10) as an equation for γ_n. The limit $\beta\to\infty$ can then be readily taken using the relation

$$\lim_{\beta\to\infty}\frac{1}{\beta}\tanh^{-1}[\tanh(\beta\tilde{J}/\sqrt{M})\tanh\beta u_K]=\operatorname{sgn}\tilde{J}\operatorname{sgn}u_K\min(|u_K|,|\tilde{J}|/\sqrt{M}) \tag{13}$$

where the substitution $J=\tilde{J}/\sqrt{M}$ is performed. Again we worked out the case of a $\pm J$ distribution.

In the RS case one finds

$$\gamma^M(x)=\exp\left(\frac{J^2x^2}{2}\right)\left\{1-\frac{1}{\sqrt{M}}\frac{2}{3\sqrt{2\pi}}J^2x^2+\frac{1}{M}\left[\left(\frac{1}{9\pi}-\frac{1}{12}\right)J^4x^4-\frac{2}{9\pi}J^2x^2\right]\right\} \tag{14}$$

up to $O(1/M)$ and

$$\frac{E_0}{J\sqrt{M}}=-\sqrt{\frac{2}{\pi}}\left[1-\frac{1}{\sqrt{M}}\frac{1}{\sqrt{18\pi}}+\frac{1}{M}\left(\frac{7}{12}-\frac{1}{9\pi}\right)+\ldots\right]$$
$$=-0.798+\frac{1}{\sqrt{M}}0.106-\frac{1}{M}0.437+\ldots. \tag{15}$$

In the one-step RS breaking the expressions are too long to be given here and will be given in detail elsewhere (Goldschmidt and De Dominicis 1989).

In that case $m\to0$ as $\beta\to\infty$, but the product $\gamma\equiv m\beta$ approaches a constant, which is determined by extremising the free energy. The final result for the ground-state energy density is given by

$$\frac{E_0}{J\sqrt{M}}=-0.765+\frac{1}{\sqrt{M}}0.010-\frac{1}{M}0.390+O\left(\frac{1}{M\sqrt{M}}\right). \tag{16}$$

† Notice the similarity of these expressions to those of Mézard *et al* (1986) for the infinite-ranged model.

Table 1. Various calculations of $E_0/J\sqrt{M}$ for different values of M.

M	Banavar *et al* (1987)	Mézard and Parisi (1987)	$1/\sqrt{M}$ expansion RS case (15)	One-step RS breaking (16)
9	−0.792	−0.810	−0.811	−0.805
10	−0.789	−0.809	−0.808	−0.801
11	−0.786	−0.806	−0.805	−0.797
19	−0.777		−0.797	−0.783
20	−0.776		−0.796	−0.782
∞	−0.763	−0.798	−0.798	−0.765

The leading terms coincide with the Parisi result for the infinite-ranged model in the first stage of RS breaking.

In table 1 we summarise the results from various calculations for $E_0/J\sqrt{M}$. In column 1 we give the numerical results of Banavar *et al* (1987), which fit the empiric formula ($c = 1.5266$):

$$\frac{E_0}{J\sqrt{M}} = -\frac{1}{\sqrt{M}}\frac{M+1}{2}\frac{c}{\sqrt{(M-1+c^2)}} = -0.763 - \frac{1}{M}0.256 + \ldots \tag{17}$$

in the range $2 \leqslant M \leqslant 20$, for which their graph bipartitioning simulations have been performed. In column 2 we give the results of Mézard and Parisi (1987) assuming replica symmetry and using the three δ-function solution for $P(h)$ (which does not include any continuous part). In column 3 we give our results of the $1/\sqrt{M}$ expansion in the replica symmetric case (15) and in column 4 we display our results with the one-step RS breaking equation (16). We see that the one-step RS breaking† results approach better the numerical results than the replica symmetric results and the error is about 0.8% for $M = 20$ and 1.6% for $M = 9$.

Note also that the coefficient of the $1/\sqrt{M}$ term in the one-step RS breaking (16) is 10 times smaller that the coefficient in the RS solution and it may become even smaller as more steps of breaking are introduced, thus approaching closer to the empiric formula of Banavar *et al.* We mention that in the fixed connectivity case the cost function C of the graph bipartitioning problem is expected to relate to the spin-glass ground-state energy density E_0 via $C/N = (M+1)/4 + E_0/2J$, where N is the number of sites in the graph and $M+1$ is the fixed connectivity‡.

4. Average connectivity

Finally let us mention the $1/M$ (or $1/\sqrt{M}$) expansion works equally well in the average connectivity case. In that case the kernel g_n^M in equations (1) and (10) is to be replaced by $\exp(M+1)(g_n - 1)$ where $M+1 = \alpha$ is the average connectivity. The free energy in (6) is to be replaced by equation (3) of Mottishaw and De Dominicis (1987).

† Previous work on RS breaking in this model has been done by Wong and Sherrington (1988) but they considered only the case of low M (2 and 3) with no overlap between the different states of the system, unlike our approach.

‡ This relation between the graph partitioning problem and the spin-glass problem has been demonstrated in the infinite connectivity case by Fu and Anderson (1986). For finite *fixed* connectivity it has been argued by Mézard and Parisi (1987) on the basis of the expectation that the effective field distribution is even when $M+1 > 2\ln 2$.

The calculations discussed in this paper can also be extended to the Potts spin glass (Goldschmidt 1988a, b, Goldschmidt and Lai 1988) which relates to the problems of graph q-partitioning and colouring (Lai and Goldschmidt 1987).

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